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INSTABILITY WAVES AND THE ORIGIN OF BUBBLES IN FLUIDIZED BEDS—II

COMPARISON WITH THEORY

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Abstract—The two-fluid equations describing the motions of isothermal beds are discussed from a continuum mechanics viewpoint. The questions of the invariant form of the relative phase acceleration, the proper description of the "viscosity" of fluidized systems, and the nature of the individual phase pressures are discussed. The linear instability wave problem is formulated and solved and the theory is shown to be capable of describing wave propagation properties measured in an earlier experimental study. The data are then used to deduce values of material constants which are both internally consistent and of reasonable magnitude.

1. INTRODUCTION

In Part I, (El-Kaissy & Homsy 1976), hereafter referred to as I, we published the results of a study of the growth and propagation of instability waves in a fluidized bed. That study had as objectives the gathering of data of sufficiently high quality for comparison with theory, and the exploration of the connection, if any, between waves and bubble origin. The latter objective was resolved in I in the affirmative. It is the comparison with theory which is the subject of the present paper.

There is only one previous study, Anderson & Jackson (1968, 1969), which had similar objectives. However, the experimental data reported by them were in part obtained visually, and all data analysis was done by hand, whereas in I we employed a minicomputer system capable of performing the averaging necessary to ensure statistical significance. A detailed comparison of their data with ours shows them to be not statistically significant. Improper averaging leads to a growth in amplitude with height which is linear, but illusory (El-Kaissy 1975). This is evident from the "adjustment" applied to the wave amplitudes in order to obtain the expected exponential growth in distance. They also include, in the theory, a horizontal structure for disturbances which is unsupported by experimental observation. Thus, any direct comparison of their work with ours is difficult. As will develop below, comparison of theory and experiment for the wave problem is by no means straightforward. Briefly, this results from the fact that in order to obtain good data on the organized wave motions before they become chaotic, it is necessary to use liquid fluidized beds. This in turn necessitates the inclusion, in the theory, of important physical effects (specifically, virtual mass terms and a "collisional pressure" term), which are believed to be relatively unimportant in particles fluidized by gas.

In order to develop a continuum description, some preliminary steps are necessary. In Section 2, we present a partial discussion of the so-called two fluid equations and the continuum hypothesis which underlies them. We highlight the axiomatic continuum mechanical development that can be used to close these equations. We thus arrive at a relatively simple set of equations which have been developed by others using different methods. Sections 3 and 4 contain the details of the solution of the linear instability problem and comparison with experiment.

2. CONTINUUM HYPOTHESIS AND THE TWO FLUID EQUATIONS

In virtually all of the theoretical work in fluidization, there emerges the idea, sometimes not explicitly stated, that the motions of a fluid-solid system, when viewed on a *continuum scale*, are often organized and in some cases may be considered deterministic. The continuum scale is clearly one which is sufficiently large to contain many elements of both continuous and

dispersed phases, but remains small in comparison with the length scale of the confining apparatus or the length scale of the phenomena. The determinism is exemplified by the bubble problem, in which many researchers have sought to elucidate the unique relationship between bubble size and bubble velocity. Given this hypothesis, one then seeks to describe these continuum properties by a suitable set of partial differential equations. Systematic derivation of these equations has been attempted in the past by the technique of volume-averaging of the point equations. For detailed accounts see Anderson & Jackson (1967), Drew & Segel (1971) and Ishii (1975). This approach is instructive because of its appealing physical significance; certain terms in the averaged equations have unambiguous origin and physical interpretation. In complicated dispersed two phase flows where the individual phase velocities are not equal, nor the concentration of one of the phases small, volume averaging looses much of its appeal. As in the averaging of any set of non-linear field equations, the averaging process produces Reynolds stress-like terms which ultimately must be modeled by some closure technique. It is only for moderately dilute systems, systems with no slip velocity, or systems in which the underlying point equations are essentially linear, that any hope for a satisfying fundamental treatment is justified (see Batchelor 1974).

The alternative approach is to proceed axiomatically, using the techniques of continuum mechanics. This tack cannot succeed where the previous one failed; however, we have found the large and well-studied literature on mixture theory to be beneficial in dealing with the two-fluid equations. Although mixture theory was originally developed to study and generalize certain results in diffusion theory for separate chemical species within the same phase, it is equally applicable to dispersed two-phase systems, given certain hypotheses. For those interested in lucid accounts of mixture theory, see especially Müller (1968), Green & Naghdi (1969) and Bowen (1971). We only give an abbreviated discussion here.

In what follows, the summation convention of repeated subscripts is in effect; the superscript refers to the phase, $a = 1,2$, for fluid and solids respectively. Where it is helpful, we make this more explicit by use of the superscripts f and s. When used as subscripts, the letters s and f refer to properties of the separate phase, not to continuum properties.

We ascribe the following properties to the mixture.

(1) Each phase has associated with it an average velocity, v_i^a.

(2) Each phase has associated with it a continuum density ρ^a. For isothermal fluidization, the continuum densities are simply related to the individual phase densities and the voidage, ϵ;

$$\rho^1 = \rho_f \epsilon; \quad \rho^2 = \rho_s(1-\epsilon). \tag{2.1}$$

(3) We identify material volumes, moving with either of the individual phase velocities, over which fundamental balances of mass, linear momentum and angular momentum may be written. Assumption of certain well-known smoothness propetries then allows the development of the differential form for each of these balances.

We do not belabor this development, but give the results. For non-reactive particles in which no phase charge occurs, the mass balances in the mixture read

$$\frac{\partial \rho^a}{\partial t} + \frac{\partial(\rho^a v_k^a)}{\partial x_k} = 0. \tag{2.2}$$

The balance of linear momentum reads:

$$\rho^a \left[\frac{\partial v_i^a}{\partial t} + v_j^a \frac{\partial v_i^a}{\partial x_j}\right] = \frac{\partial T_{ij}^a}{\partial x_j} + I_i^a + \rho^a b_i^a \tag{2.3}$$

where T_{ij}^a is the partial stress tensor giving the surface force acting on the ath constituent due

to the action of all constituents, I_i^a is the interaction force between the ath constituent and all other constituents, b_i^a is the externally applied body force per unit mass.

We do not expect a resultant force to be present in the mixture; we thus require

$$\sum_a I_i^a = 0. \qquad [2.4]$$

Equation [2.4] is merely a statement of Newton's third law.

Consideration of the balance of angular momentum is customary at this point of development, but we will not give the details here. They are contained in El-Kaissy (1975). Suffice to say that it is possible to prove that if no body couples act on the phases, individually or on both phases together, then the partial stresses are symmetric. Thus we have

$$T_{ij}^a = T_{ji}^a \, (a = 1,2) \qquad (2.5]$$

as a statement of balance of angular momentum. Equations [2.2], [2.3] and [2.5] are the final balance equations for the mixture.

We will consider only two phases ($a = 1,2$) and that the only body force acting on the system is gravitational. We also simplify the notation somewhat. Whenever it is convenient, we drop the superscripts on velocities, and use v_i for the solids velocity and u_i for the fluid velocity.

The body force terms may be written directly:

$$\rho^1 b_i^1 = -\rho_f \epsilon g \, \delta_{i3} \qquad [2.6a]$$

$$\rho^2 b_i^2 = -\rho_s (1-\epsilon) g \, \delta_{i3}. \qquad [2.6b]$$

It now remains to write constitutive equations for I_i^a, T_{ij}^a ($a = 1,2$). Modern nonlinear continuum mechanics provide elegant and systematic means for arriving at constitutive relations which are formally correct. The problem with such an approach, however, is that the generality of the results obtained restricts their use owing to the large number of phenomenological coefficients which arise. In what follows we temper the formal generality of the theory with physically-based arguments. In particular, we require the constitutive equations to satisfy the principle of material frame indifference, but we do not apply the so-called "principle" of equipresence. For a criticism of the principle of equipresence (see Rivlin 1970, 1972). Furthermore, we allow a dependence of the interaction force on interphase pressure, relative velocity, and relative acceleration. For the stresses, we allow for purely fluid-like behavior, with partial stresses related only to rates of deformation.

It is a result from the literature on mixture theory that the following quantities are frame indifferent:

$$v_i^{12} = u_i - v_i \qquad [2.7a]$$

$$d_{ij}^1 = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad [2.7b]$$

$$d_{ij}^2 = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad [2.7c]$$

$$\Omega_{ij}^{12} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) - \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad [2.7d]$$

$$\dot{v}_i^{12} = \left(\frac{\partial u_i}{\partial t} + (u_j - v_j)\frac{\partial u_i}{\partial x_j}\right) - \left(\frac{\partial v_i}{\partial t} + (v_j - u_j)\frac{\partial v_i}{\partial x_j}\right) \qquad [2.7e]$$

v_i^{12} is the relative velocity, d_{ij}^a is the rate of deformation tensor of the ath phase, Ω_{ij}^{12} is the relative spin tensor, and $\dot{v}_i^{12}$ is the relative acceleration vector. The properly invariant form of the relative phase acceleration was given implicitely by Adkins (1963) and explicitely by Drew & Segel (1971).†

As a first step in modeling multiphase systems, we propose a class of materials in which the functions I_i^a and T_{ij}^a are linear in the constitutive variables. Of course it is well known that fluid–particle systems can exhibit non-linear materials behavior (see Krieger 1972 and Jeffery & Acrivos 1976 for recent reviews). These effects seem to be second order for nearly isotropic particles, but can become quite important for fluid–fluid dispersions and particles of irregular or specific non-isotropic configurations. In any event, it is clear that all fluid–particle dispersions will behave linearly for suitably small rates of strain and relative phase velocities. Thus we write linear constitutive relations, using the fact that T_{ij}^a is symmetric,

$$I_i^1 = -p_f \frac{\partial(1-\epsilon)}{\partial x_i} - \alpha_3 v_i^{12} - \alpha_4 \dot{v}_i^{12} \tag{2.8a}$$

$$I_i^2 = -I_i^1 \tag{2.8b}$$

$$T_{ij}^1 = -p_f \epsilon \delta_{ij} + \sum_{\gamma=1}^{2} \left(\beta^{\gamma 1} d_{nn}^{\gamma} \delta_{ij} + 2\eta^{\gamma 1} \left(d_{ij}^{\gamma} - \frac{d_{kk}^{\gamma}}{3} \delta_{ij} \right) \right) \tag{2.8c}$$

$$T_{ij}^2 = -p_s(1-\epsilon)\delta_{ij} + \sum_{\gamma=1}^{2} \left(\beta^{\gamma 2} d_{nn}^{\gamma} \delta_{ij} + 2n^{\gamma 2} \left(d_{ij}^{\gamma} - \frac{d_{kk}^{\gamma}}{3} \delta_{ij} \right) \right). \tag{2.8d}$$

Drag and virtual mass effects are reflected in the coefficients α_3, α_4 respectively. The appropriate form for the interphase pressure that we have adopted here has been used earlier by Drew & Segel (1971) and Ishii (1975). The individual phases are given only fluid-like behavior, with d_{ij}^a and an isotropic term being the only variables on which the partial stresses depend. Since the isotropic term gives the pressure measured over an element of volume of the mixture and not of the single phase, we chose the above form of the isotropic pressure.

Now the remaining material functions, $\beta^{\gamma a}$, $\eta^{\gamma a}$ in general depend upon any scalar invariant of any objective quantity, e.g. $\|v_i^{12}\|^2 = v_i^{12} v_i^{12}$, $d_{ij}^a d_{ij}^a$, etc. This dependence will necessarily lead to non-linear material response. Since the application under discussion is a linear instability problem (see Section 3), it suffices to treat them as functions of the scalar variables, i.e. ρ_s, ρ_f, ϵ. The fact that the functions depend solely upon the lowest order statistic of the dispersion, namely, the voidage or the volume concentration of particles, apparently limits the linear model to statistically random dispersions. (For a discussion of this point, see Batchelor 1974 and Hashin 1964).

Insertion of these constitutive equations into the balance laws leads, after some manipulation, to the following two momentum equations.

Solid

$$\begin{aligned} \rho_s(1-\epsilon)\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] = \rho_f(1-\epsilon)\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] \\ - \frac{\partial(1-\epsilon)(p_s - p_f)}{\partial x_j} \delta_{ij} + \frac{\alpha_3}{\epsilon}(u_i - v_i) \\ + \frac{\alpha_4}{\epsilon}\left(\left(\frac{\partial u_i}{\partial t} + (u_j - v_j)\frac{\partial u_i}{\partial x_j}\right) - \left(\frac{\partial v_i}{\partial t} + (v_j - u_j)\frac{\partial v_i}{\partial x_j}\right)\right) \\ - (\rho_s - \rho_f)(1-\epsilon) g \delta_{i3} \end{aligned}$$

†Although their derivation is in error, their final result is correct.

$$+\frac{\partial}{\partial x_j}\sum_{\gamma=1}^{2}\left(\beta^{\gamma 2}d^{\gamma}_{nn}\delta_{ij}+2\eta^{\gamma 2}\left(d^{\gamma}_{ij}-\frac{d^{\gamma}_{kk}}{3}\delta_{ij}\right)\right)$$

$$-\frac{(1-\epsilon)}{\epsilon}\frac{\partial}{\partial x_j}\sum_{\gamma=1}^{2}\left(\beta^{\gamma 1}d^{\gamma}_{nn}\delta_{ij}+2\eta^{\gamma 1}\left(d^{\gamma}_{ij}-\frac{d^{\gamma}_{kk}}{3}\delta_{ij}\right)\right). \qquad [2.9a]$$

Fluid

$$\rho_f\epsilon\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\epsilon\frac{\partial p_f}{\partial x_j}\delta_{ij}-\alpha_3(u_i-v_i)$$

$$-\alpha_4\left(\left(\frac{\partial u_i}{\partial t}+(u_j-v_j)\frac{\partial u_i}{\partial x_j}\right)-\left(\frac{\partial v_i}{\partial t}+(u_j-v_j)\frac{\partial v_i}{\partial x_j}\right)\right)$$

$$+\frac{\partial}{\partial x_j}\left(\sum_{\gamma=1}^{2}\beta^{\gamma 1}d^{\gamma}_{nn}\delta_{ij}+2\sum_{\gamma=1}^{2}\eta^{\gamma 1}\left(d^{\gamma}_{ij}-\frac{d^{\gamma}_{kk}}{3}\delta_{ij}\right)\right)-\rho_f g\epsilon\delta_{i3}. \qquad [2.9b]$$

We wish to point out that the basic set of equations is not closed, even if all the material constants are specified, provided one treats the pressures p_s, p_f as dependent variables. Indeed there are 9 unknowns (u_i, v_i, ϵ, p_f, p_s) but only 8 equations. This has been noted before (Anderson & Jackson 1968, Murray 1965). For isothermal fluidization, it is appealing to look for analogies in single phase fluid mechanics, in which the pressure may be either taken, in the compressible case, as a function of density; or treated, in the incompressible case, as a function allowing the solenoidal constraint to be satisfied. The analogy must not be taken literally however, since specification of both p_f and p_s leads to an overdetermined system. It is common in the literature on two phase (gas–liquid) systems to assume $p_f = p_s$ (perhaps with a jump to account for surface tension in bubbly flows). This assumption is also common in the literature on gas fluidized systems. Although it is valid at the minimum fluidization condition, it breaks down as the collisional momentum transport on the microscale becomes appreciable with increase of fluid phase velocity beyond the minimum fluidization velocity. Effects of collisions can be represented by the term $p_s - p_f$, the "collisional pressure", as discussed by Drew & Segel (1971) and are expected to be a strongly decreasing function of voidage. So in order to close the basic set of equations, we assume $p_s - p_f$ the "collision pressure" to be a function of continuum density, i.e. $p_s - p_f = f(\epsilon)$.

It has been noted by many authors that the field equations have a particularly simple solution, referred to as homogeneous fluidization. This state is obtained by seeking solutions of the form

$$\epsilon = \epsilon_0 \quad \text{(a constant)}$$
$$u_i = u_0\,\delta_{i3} \qquad [2.10]$$
$$v_i = (0,0,0)^T.$$

The continuity equations are trivially satisfied, and the momentum equations reduce to

$$\frac{\partial p_f}{\partial x_i} = -\frac{\alpha_3}{\epsilon_0}u_0\,\delta_{i3} - \rho_f g\,\delta_{i3}$$

$$\frac{(1-\epsilon_0)\,\partial(p_s-p_f)}{\partial x_i} = \frac{\alpha_3}{\epsilon_0}u_0\,\delta_{i3} - (1-\epsilon_0)(\rho_s-\rho_f)g\,\delta_{i3}. \qquad [2.11]$$

Consider the condition of minimum fluidization. This condition is defined as that velocity for

which the drag on the particle phase just balances the net weight of the particles. With this definition,

$$p_s = p_f, \quad \alpha_3 u_{mf} = (1-\epsilon_{mf})(\rho_s - \rho_f)g,$$

or solving for u_{mf},

$$u_{mf} = \frac{(1-\epsilon_{mf})(\rho_s - \rho_f)g}{\alpha_3(\epsilon_{mf})}. \tag{2.12}$$

Provided the voidage at incipient conditions may be estimated, [2.12] then provides the basis of correlations for u_{mf} for small particles (Davidson & Harrison 1963, pp. 13–14).

For $u_i > u_{mf}\delta_{i3}$, fluidization occurs. It is clear from the preceeding discussion that there exists a stationary solution to the equations with $\epsilon \geq \epsilon_{mf}$, p_f being determined from Archimedian hydrostatics, ϵ being determined from [2.12], and $v_i = 0$. This solution has in the past been denoted "homogeneous" fluidization. While it constitutes a solution to the problem, it is well known that it is unstable with respect to other, more complicated motions. We can develop this solution a bit further. It is a result of experiments that dp_f/dx_3 remains constant above u_{mf}. Thus $\epsilon_0 > \epsilon_{mf}$, i.e. the bed expands. In order to find a solution consistent with these observations, we must have

$$\frac{dp_f}{dx_3} = -(1-\epsilon_o)\rho_s g - \epsilon_o \rho_f g \tag{2.13a}$$

$$\frac{\alpha_3(\epsilon_o)}{\epsilon_o} u_o = (1-\epsilon_o)(\rho_s - \rho_f)g. \tag{2.13b}$$

It is well-known that (α_3/ϵ) is a monotonically decreasing function of ϵ, so a solution of [2.13b] for $u_o - u_{mf}$ consistent with the observed experimental behavior that $dp_f/dx_3 \sim$ const., $\epsilon_o > \epsilon_{mf}$ is possible.

3. THE LINEAR INSTABILITY PROBLEM

As we have stated, the primary purpose of this paper is to explore the ability of the continuum equations, with the closure [2.8] to *quantitatively* describe the features of the experiments reported in I. The primary quantities measured therein were the frequency, f, velocity of propagation, V, and growth constant, σ_r of linearly unstable modes of motion which occur for $u = u_o > u_{mf}$. We first develop the equations in dimensionless form, make some simplifications, and then discuss the linear instability predictions. It proves convenient to make the equations dimensionless using d_p and u_o as length and velocity scales respectively. To avoid confusion, all velocities (including the scaling velocity u_o) are *intersticial* velocities appropriate to the theoretical development of section 2. Thus, denoting dimensionless quantities by primes,

$$x_i' = x_i/d_p, \quad u_i' = \frac{u_i}{u_o}, \quad v_i' = \frac{v_i}{u_o}, \quad t' = \frac{tu_o}{d_p}, \quad p_a' = \frac{p_a}{\rho_s u_o^2}. \tag{3.1}$$

The equations, in their dimensionless form, then become (dropping the primes),

Continuity

$$\left.\begin{aligned} &\textit{Fluid:} \quad \frac{\partial \epsilon}{\partial t} + \frac{\partial}{\partial x_k}(\epsilon u_k) = 0 \\ &\text{Solid:} \quad \frac{\partial}{\partial t}(1-\epsilon) + \frac{\partial}{\partial x_k}[(1-\epsilon)v_k] = 0. \end{aligned}\right\} \tag{3.2}$$

Momentum

$$\text{Solid:}\quad (1-\epsilon)\left[\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right]=R/Re\left\{\frac{\partial}{\partial x_i}\left(\sum_{\gamma=1}^{2}\hat{\beta}^{\gamma 2}\mathrm{d}_{nn}^{\gamma}\delta_{ij}\right.\right.$$
$$+2\dot{\eta}^{\gamma 2}\left(d_{ij}^{\gamma}-\frac{d_{kk}^{\gamma}}{3}\delta_{ij}\right)\Big)-\frac{(1-\epsilon)}{\epsilon}\frac{\partial}{\partial x_j}\left(\sum_{\gamma=1}^{2}\hat{\beta}^{\gamma 1}d_{nn}^{\gamma}\delta_{ij}\right.$$
$$\left.\left.+2\dot{\eta}^{\gamma 1}\left(d_{ij}^{\gamma}-\frac{d_{kk}^{\gamma}}{3}\delta_{ij}\right)\right)+\hat{\alpha}_3(u_i-v_i)\right\} \qquad [3.3]$$
$$+R(1-\epsilon)\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]-\frac{\partial[(1-\epsilon)(p_s-p_f)]}{\partial x_i}$$
$$+R\hat{\alpha}_4\left(\left(\frac{\partial u_i}{\partial t}+(u_j-v_j)\frac{\partial u_i}{\partial x_j}\right)-\left(\frac{\partial v_i}{\partial t}+(v_j-u_j)\frac{\partial v_i}{\partial x_j}\right)\right)-(1-\epsilon)(1-R)\mathrm{Fr}\,\delta_{i3}\,.$$

$$\text{Fluid:}\quad R\epsilon\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=-\epsilon\frac{\partial p_f}{\partial x_i}+\frac{R}{Re}\left\{\left(\frac{\partial}{\partial x_i}\hat{\beta}^{11}\frac{\partial u_k}{\partial x_k}\right)\right.$$
$$+\frac{\partial}{\partial x_j}\left(\hat{\eta}^{11}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\right)+\frac{\partial}{\partial x_i}\left(\hat{\beta}^{21}\frac{\partial v_k}{\partial x_k}\right)$$
$$\left.+\frac{\partial}{\partial x_j}\left(\eta^{21}\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right]\right)-\hat{\alpha}_3\epsilon(u_i-v_i)\right\}$$
$$-\epsilon\hat{\alpha}_4R\left(\left(\frac{\partial u_i}{\partial t}+(u_j-v_j)\frac{\partial u_i}{\partial x_j}\right)-\left(\frac{\partial v_i}{\partial t}+(v_j-u_j)\frac{\partial v_i}{\partial x_j}\right)\right)-R\epsilon\mathrm{Fr}\,\delta_{i3}\,. \qquad [3.4]$$

In writing these equations, we have introduced the dimensionless material constants

$$\hat{\eta}^{ij}=\eta^{ij}/\mu_f$$
$$\hat{\beta}^{ij}=\beta^{ij}/\mu_f$$
$$\hat{o}_3=\alpha_3\,d_p^{\,2}/\epsilon\mu_f$$
$$\hat{\alpha}_4=\alpha_4/\epsilon\rho_f\,. \qquad [3.5]$$

In addition, we find the following dimensionless parameters:

ϵ_0 void fraction

$R\equiv\dfrac{\rho_f}{\rho_s}$ density ratio

$Re\equiv\dfrac{u_o d_p\rho_f}{\mu_f}$ particle Reynolds number

$\mathrm{Fr}\equiv gd_p/u_o^{\,2}$ particle Froude number.

We emphasize here that the parameters in [3.6] are easily calculated, being related to macroscopic quantities, while the material constants in [3.5] must be measured experimentally.

As we have already noted, these equations possess a simple steady solution, known as the state of "homogeneous" fluidization. It is given dimensionlessly as

$$u_j=\delta_{3j}$$
$$v_j=0$$
$$\epsilon=\epsilon_0 \qquad [3.7]$$

provided that

$$\hat{\alpha}_3(\epsilon_0) = \frac{Re(1-\epsilon_o)(1-R)\mathrm{Fr}}{R}. \qquad [3.8]$$

Equation [3.8] is a dimensionless statement of the fact that the drag induced by the uniform flow u_j is balanced by the net weight of the particles. Equation [3.8] provides a basis for measurement of the constant $\hat{\alpha}_3(\epsilon_0)$. Time-averaged expansion data, giving ϵ_0 vs u_0, are sufficient to fit the function $\hat{\alpha}_3$ to experiment, provided that any motions (particle *or* fluid) are small amplitude and have zero time average. In I, we reported the results of expansion data, and they were seen to be in good agreement with the Richardson–Zaki correlation, which in the present variables reads

$$\hat{\alpha}_3 = \frac{Re(1-\epsilon)(1-R)\mathrm{Fr}}{R}\left(\frac{u}{u_t}\right)\epsilon^{(1-n)} \qquad [3.9]$$

where n is the Richardson–Zaki exponent, and u_t is the terminal velocity of a single sphere, obtainable from correlations once Re is given. For reference, this relation has been reduced to numerical form for the conditions of the experiments and is given in table 1. (The designations A, B, C, D refer to bead sizes and conditions of I.)

We wish to examine the stability of the steady solution [3.7]–[3.8] to small disturbances. It is common in linear instability theory to suppose that the steady solution [3.7]–[3.8] is subject to small perturbations, viz.

$$\begin{aligned} u_j &= \delta_{3j} + u_j' \\ v_j &= v_j' \\ \epsilon &= \epsilon_0 + \epsilon' \end{aligned} \qquad [3.10]$$

and equations for the primed variables may be obtained by substitution into [3.1]–[3.4] making use of the base state [3.7]–[3.8]. The full non-linear equations are complicated and serve no purpose here, so we present here their linearized form.

Experiments have shown that over some distance above the distributor, the motion is planar and one-dimensional. It is this regime we wish to analyze. We thus take all variables to be independent of x_1 and write for simplicity

$$v_i'(t,x_3) = v(t,z); \quad u_i'(t,x_3) = u(t,z); \quad \epsilon_i'(t,x_3) = \epsilon_i(t,z).$$

Table 1. The values of $\hat{\alpha}_3(\epsilon)$ obtained from expansion data

Set	$\hat{\alpha}_3$
A	$78\ (1-\varepsilon)\varepsilon^{-1.87}$
B	$109\ (1-\varepsilon)\varepsilon^{-1.65}$
C	$146\ (1-\varepsilon)\varepsilon^{-1.44}$
D	$212\ (1-\varepsilon)\varepsilon^{-1.34}$

In this case [3.2]–[3.3] are a closed system for (u, v, ϵ'); the one-dimensional linear version of [3.4] serves to determine the fluid pressure, p_f. Substitution of these forms leads to substantial simplifications. We define a vector as

$$\psi = \begin{pmatrix} u \\ v \\ \epsilon' \end{pmatrix} \qquad [3.11]$$

and the linear stability equations may be written compactly as

$$A\psi = 0 \qquad [3.12]$$

where A is a vector differential operator. In particular, the explicit form of A is

$$A = \left\{ \begin{array}{c|c|c} \epsilon_0 \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial z} \\ \hline 0 & (1-\epsilon_0)\dfrac{\partial}{\partial z} & -\dfrac{\partial}{\partial t} \\ \hline \dfrac{R}{Re}\left[\xi^{12}\dfrac{\partial^2}{\partial z^2} + \hat{\alpha}_3{}^0\right] & \dfrac{R}{Re}\left[\xi^{22}\dfrac{\partial^2}{\partial z^2} - \hat{\alpha}_3{}^0\right] & M\dfrac{\partial}{\partial z} + \dfrac{R}{Re}\dfrac{d\hat{\alpha}_3}{d\epsilon}\Big|_{\epsilon_0} \\ + R\hat{\alpha}_4{}^0\left[\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial z}\right] & - R\alpha_4{}^0\left[\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial z}\right] & + (1-R)\mathrm{Fr} \\ + R(1-\epsilon_0)\dfrac{\partial}{\partial t} & -(1-\epsilon_0)\dfrac{\partial}{\partial t} & \end{array} \right\} \qquad [3.13]$$

Here we have put $\xi^{12} = \hat{\beta}^1 + \frac{4}{3}\hat{\eta}^1$; $\xi^{22} = \hat{\beta}^2 + \frac{4}{3}\hat{\eta}^2$; and

$$M = -(1-\epsilon_0)\frac{\partial(p_s - p_f)}{\partial\epsilon}\Big|_{\epsilon=\epsilon_0}, \text{ where}$$

$$\hat{\beta}^\gamma = \hat{\beta}^{0,\gamma} - \frac{(1-\epsilon_0)}{\epsilon_0}\hat{\beta}^{0,\gamma}$$

$$\hat{\eta}^\gamma = \hat{\eta}^{0,\gamma} - \frac{(1-\epsilon_0)}{\epsilon_0}\hat{\eta}^{0,\gamma}.$$

In these equations the superscript denotes a material constant evaluated at the base state, e.g. $\hat{\alpha}_3{}^0 = \hat{\alpha}_3(\epsilon_0)$, etc. Note that we have set $(p_s - p_f) \neq 0$, and consider $(\partial(p_s - p_f)/\partial\epsilon)$ as a quantity to be specified. We note that rheological material constants appear only in the combination of $(\beta + 4/3\mu)$, as is well known for compressional motion of the form $u_3 = u_3(x_3, t)$, etc. The linear instability problem may therefore be viewed as the locus of null vectors ψ^* of the operator A. We seek solutions of the form

$$\psi^* = \psi^\circ \exp(\sigma t + ik_z z) \qquad [3.14]$$

where σ, k_z are linked together by the requirement that

$$\det(A^*) = 0. \qquad [3.15]$$

A^* is the matrix obtained from [3.13] by the substitution $(\partial/\partial t) = \sigma$ and $(\partial/\partial z) = ik_z$.

Equation [3.15] then gives a complex-valued polynomial whose roots give the dispersion relation.

For the disturbance motion growing in time, σ is complex, $\sigma = \sigma_r + i\sigma_i$, k_z is real, and it is common to write $\sigma = -ik_z c$ resulting in the usual form $\exp[ik_z(z-ct)]$. k_z is the wave number, related to the wave length $\lambda = 2\pi/k_z$, the wave velocity in the vertical direction is given as $v = c_r = (-\sigma_i/k_z)$ and the growth constant is $\sigma_r = k_z c_i$.

The analysis for growth in time assumes a disturbance to posess periodic spatial behavior while growing exponentially in time. However, what one observes experimentally is a spatially periodic progressive wave train whose amplitude grows with distance. We can make the connection between the two by using the results of Gaster (1963), which show that, *for small amplification rates*, the frequency of the spatially and temporally growing modes are equal, and the growth constants are related through the group velocity. Since the waves are only weakly dispersive, and the comparison with experiment is subject to a fair degree of subjectivity, we have simply used the phase velocity in making the comparison between the temporal predictions and experiment.

4. COMPARISON WITH EXPERIMENT

The theory is capable of making predictions of the wave speed and growth constant as a function of wave number, provided the material functions M, $\hat{\alpha}_4$, ξ^{12}, ξ^{22} are given. There is little previous experimental or theoretical evidence on any of these although it is possible to infer some expected magnitudes. A small amount of information may be obtained regarding the virtual mass coefficient, α_4. It is well-known that the virtual mass coefficient for a single sphere may be obtained by solution of the potential flow equations (see, e.g. Landau & Lifshitz 1959, section 24). If one assumes this is true for assemblages and adopts a so-called "cell-model", one finds Lamb (1945, section 93)

$$\hat{\alpha}_4 = \frac{3+\epsilon}{2\epsilon}. \qquad [4.1]$$

We therefore expect values of $\hat{\alpha}_4$ on the order of 3.5 for the conditions of our experiments.

Not much is known about the bulk viscosities β^{22}, β^{21}, β^{12}, β^{11} and the shear viscosities η^{22}, η^{12}, η^{21}, η^{11} appearing in [2.14]. The shear viscosity of the fluidized beds may be as high as 10–20 poise, and certainly decreases strongly with increasing voidage (see, e.g. Davidson *et al.* 1977): that most of the momentum transport is due to collisional processes involving particles and their resistance to sliding (see Frankel & Acrivos 1967 and Murray 1967). We adopt the point of view taken by many authors, and discussed in some detail by Davidson *et al.* (1977), that the rheological coefficient η^{12} associated with resistance to deformation of the fluid phase is small compared to the coefficient η^{22} associated with that of the solid phase. This is admittedly an assumption, but it is plausible under the conditions that the void fraction is small (dense systems) and that the resistance to transport offered by particle interactions is much greater than that due to the deformations of the continuous phase. This is undoubtedly true in dense granular materials and gas fluidized systems: the justification most often cited is the early work by Bagnold (1954), but some recent work by Savage (1979) on granular flow provides further support for this approximation. We also expect β^{21} to be small compared with β^{22}. Finally, there are no extimates available for the parameter M which relates to dependence of collisional pressure on voidage.

In developing predictions, we therefore adopt the following range of parameters as being reasonable

$$\hat{\alpha}_4 \sim 0(1\text{–}10) \qquad [4.2a]$$

$$\xi^{12} \ll \xi^{22} \qquad [4.2b]$$

$$\xi^{22} \sim 0 \qquad [4.2c]$$

$$M \sim 0(1\text{–}10) \qquad [4.2d]$$

and proceed to generate numerical results for values of R, Re, Fr, $\hat{\alpha}_3$ appropriate to the experiments.

Figure 1 gives a typical set of predictions for the conditions of Run $B1$ in I. The general features are that the waves are predicted to be only weakly dispersive and that there exists a wave number of maximum growth rate. This presumably corresponds to wave number observed experimentally.

Our approach to the comparison of theory with experiment was as follows: the four material constants were systematically varied over reasonable ranges until the three measured wave properties, namely the wave number, the growth constant and the velocity of propagation, matched the predictions to within experimental error. This is thus a parameter estimation problem and, because of the large experimental variation of some of the properties within one experiment, is a difficult one. However as it will develop below, the range of model parameters necessary to span the experimental variation is surprisingly small. Furthermore, all predictions were insensitive to the choice of ξ^{12}, as long as [4.2b] was satisfied. Thus we are left with three material constants with which to fit three experimental variables. To include more material parameters or to refine the estimates given below would require experimental knowledge of at least part of the dispersion relation. These experimental data are unavailable at present.

Table 2 gives the results of the fits of theory to experiment. Variations in experimental quantities are given. When two or more choices of material constants fit the experiments, we give them. We have presented the results of I in *dimensionless* form, using the scalings in [3.1].

Generally, it is seen that there is a trade-off between ξ^{22} and $(\hat{\alpha}_4, M)$. A fit is sometimes possible for both large and small values of these parameters. Other than that, the results in table 2 are quite encouraging. In particular it is possible to fit the experiments with values of material constants which are both reasonable in magnitude and consistent between sets of data for which both the particle size and fluidizing velocity were varied over a relatively wide range. Although we are tempted to go one step further and deduce the dependence of material constants on voidage from the data in table 2, we do not feel that the experiments are sufficiently accurate to justify doing so. Recalling however that the sets of data are numbered in order of increasing particle size and increasing voidage within each set, we find a consistency in table 2 in the sense that

(1) ξ^{22} decreases with increasing voidage.

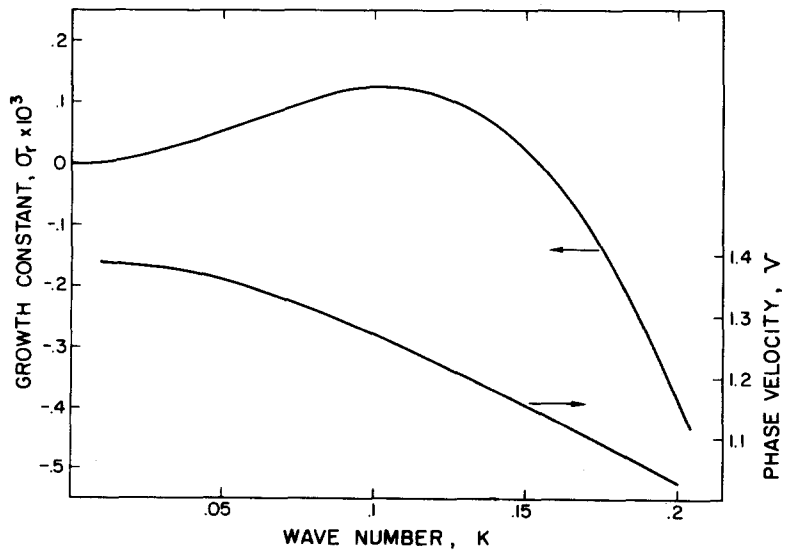

Figure 1. Typical linear instability predictions for the conditions of Run $B1$. $\xi^{22} = 6.5 \times 10^3$, $\hat{\alpha}_4 = 5.0$, $M = 7.0$.

(2) $M = -(1-\epsilon_0)\,(\partial(p_s - p_f)/\partial\epsilon)|_{\epsilon=\epsilon_0}$ is always positive implying that the collisional pressure always decreases with increasing voidage.

We carried out a sensitivity analysis to see how the propagation properties change with small changes in the values of the deduced material constants given in table 2. We can write the change in growth constant as

$$d\sigma = \left(\frac{\partial\sigma}{\partial\xi^{22}}\right)_{\hat{\alpha}_4,M,k_z} d\xi^{22} + \left(\frac{\partial\sigma}{\partial\hat{\alpha}_4}\right)_{M,k_z,\xi^{22}} d\hat{\alpha}_4 + \left(\frac{\partial\sigma}{\partial M}\right)_{\hat{\alpha}_4,k_z,\xi^{22}} dM. \qquad [4.3]$$

Table 2. Comparison of theory and experiment

Run	k	$\sigma_r \times 10^3$	V	$\xi^{22} \times 10^{-3}$	$\hat{\alpha}_4$	M
A1	.07-.157	.4	1.48	--	experiment	--
	.125	.417	1.46	5.5	3	7
	.1	.4	1.44	9.5	4	8
	.125	.422	1.41	7	3	5
A2	.2	2.7	1.5	--	experiment	--
	.2	2.7	1.32	2.5	6	9
	.2	2.6	1.32	3.5	3	5
A3	.17-.25	4.4	1.45-1.72	--	experiment	--
	.2	4.6	1.39	1.5	11	17
	.225	4.35	1.36	1.5	5	8
B1	.062-.115	1.0	1.2-1.32	--	experiment	--
	.125	1.04	1.20	7.5	6	8
	.1	1.03	1.27	6.5	5	7
	.1	1.04	1.22	9.5	7	9
B2	.186-.279	1.83	1.26-1.35	--	experiment	--
	.2	1.93	1.23	1.5	2	4
B3	.296-.31	7.9	1.25	--	experiment	--
	.225	7.8	1.23	1.5	7	8
	.25	7.95	1.20	1.0	6	9
C1	.17-.32	1.87	.97-1.03	--	experiment	--
	.15	1.89	.968	2.0	14	13
	.15	1.79	.983	1.5	17	16
	.175	1.89	.991	1.0	10	10

Table 2. (*Contd*)

Run	k	$\sigma_r \times 10^3$	V	$\xi^{22} \times 10^{-3}$	$\hat{\alpha}_4$	M
C2,C3	.17-.323	6.6	1.0	--	experiment	--
	.225	6.81	.957	.5	12	11
	.25	6.6	.96	.4	8	10
D1	.075-.32	3.6	.92	--	experiment	--
	.2	3.56	.914	1.5	4	4
	.175	3.56	.908	2.0	8	7
	.15	3.5	.921	1.5	18	15
D2	.25-.34	13.7	.91	--	experiment	--
	.275	13.8	.88	1.0	8	6
D3	.31-.39	23.	.94	--	experiment	--
	.325	23.9	.85	.4	3	3
	.375	22.1	.86	.6	5	7

By evaluating the partial derivatives in [4.3], we found that for the same percentage change in the values of the three material constants, the contributions from the three terms on the left hand side are of the same order of magnitude. This implies that the values of *all* three material constants should be known to the same degree of accuracy for predicting the motions in a fluidized bed, at least in the case of fluidization by a liquid.

In conclusion we have found a continuum theory capable of quantitatively and consistently describing the propagation of small amplitude instability waves in fluidized beds, and we have used the experimental data to deduce values of the material constants appearing in the constitutive equation.

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